**Dov M. Gabbay** and **Karl Schlechta**
*Conditionals and Modularity in General Logics.*
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This book is not for the faint-hearted, nor for the novice. It presupposes a good knowledge of classical logic and its model-theory, and also familiarity with the preferential semantics for systems of uncertain (aka nonmonotonic) consequence. But its underlying theme may be appreciated given just a little acquaintance with each.

It is customary in philosophical discussions of logic to distinguish between syntactic and semantic levels, and logicians themselves use this terminology in their technical work. But in reality there are three conceptual levels, not two. The *syntactic* level concerns expressions of a formal (or natural) language alone, irrespective of external structures that might be used to give it meaning; the *semantic* level concerns concepts that link the language with external structures; and the *model-theoretic* level concerns the external structures themselves irrespective of formal languages that may be used to describe them. The last of these three is sometimes also referred to as the 'purely mathematical' level. Very often, as in the book under review, it is also called 'semantic', with only the context distinguishing it from the level of concepts and results that *link* formal languages with external structure.

The theme of this book is to construct, as far as possible, model-theoretic counterparts of concepts that are usually formulated on the syntactic level and to obtain purely mathematical results from which semantic and syntactic theorems, both well-known and novel, may be obtained as corollaries. Travelling along this road can have a number of advantages beyond that of giving us a view of the same mountain from a different angle: we no longer have to take account of distinct but logically equivalent formulae; we can often see variegated syntactic conditions emerging as outcomes of combining just a few model-theoretic options; and we sometimes find paths to generalization opening more naturally than they do on the syntactic level. Optimally, the purely mathematical results can also be of interest in themselves, independently of their applications to logic.

The theme is developed to cover consequence relations for uncertain inference, based on some version of the so-called preferential semantics of Shoham and others, as well as a fairly wide range of many-valued deductive logics. In this coverage it differs from the model-theory that is familiar from mainstream mathematical logic in the twentieth century, which focused almost entirely on classical first-order logic and its infinitary or higher-order extensions. Given the additional complexities in the non-classical and uncertain contexts, the authors restrict their investigations in this book to the propositional level.

It may be wondered whether this kind of enterprise could be carried out trivially. Perhaps we need only identify a formula with the set of all its models and then routinely rewrite all concepts and results about formulae in terms of those sets. Unfortunately, this simple recipe does not work well. One reason is that routine translation of syntactic material need not give us

anything illuminating. Imagine, for example, that we simply translated the recursive definition of theoremhood for some axiom system for classical logic into a correspondingly recursive definition on the model-theoretic level, and then restate and prove the completeness theorem for classical logic in terms of that translation. The enterprise would as uninformative as it is tedious. But on the other hand, illuminating model-theoretic analogues of the completeness theorem have long been known to be available—for example, the fact that every subset of a Boolean algebra with the finite intersection property may be extended to a maximal proper filter of that algebra, which can also be expressed by saying that every Boolean algebra is a subdirect product of copies of the two-element one. Another reason why the simple 'rewrite recipe' fails is that while every formula may be associated with the set of the models that satisfy it, the association need not be surjective. In other words, there may be sets $M$ of models such that for no formula (indeed for no set of formulae) is $M$ exactly the set of all models satisfying it. In that case, one says that the set of models is not definable in the logic. In the classical context, this happens as soon as the formal language has infinitely many elementary letters; for many non-classical logics, it can also arise in the finite case. In both contexts it introduces delicate and sometimes difficult problems.

Thematically, the text has three main parts. The first concerns many-valued deductive logics that have monotony or antitony properties including, as a limiting case, classical logic. The most important result established in this part of the book is that all such logics satisfy interpolation on the model-theoretic level, although not always on the syntactic level as the resources provided by their connectives may be insufficient to define the interpolating sets of models.

The second part concerns logics that have emerged from the well-known preferential semantics for uncertain inference. Here, the model structures make use of a selection function $\mu$ taking each set $M$ of classical models to a subset $\mu(M) \subseteq M$. If we are coming to this function from the notion of a preference relation between models, we may think intuitively of $\mu(M)$ as consisting of the 'most preferred', 'most normal' or 'best' models in $M$, that is, those that are minimal under whatever preference relation one has in mind. However, we may also give $\mu$ an intuitive but non-relational reading. Given a set $M$ of models, the subsets $X \subseteq M$ with $\mu(M) \subseteq X$ (which together form a principal filter over the power set of $M$) may be seen as being the subsets of $M$ that are 'large' when considered as subsets of $M$ itself. The inference relation between formulae defined as usual by putting $\varphi \mid\sim \psi$ iff $\mu(\underline{\varphi}) \subseteq \underline{\psi}$ (in this review, underlining a formula to indicate the set of its models) may then be read as requiring that a large part of the set of models satisfying $\varphi$ also satisfy $\psi$, in other words, that only a small part of $\underline{\varphi}$ lies outside $\underline{\psi}$. This is the reading that guides the authors. The most important formal result obtained in this part of the volume tells us that a certain purely mathematical 'multiplicative size rule' for $\mu$ suffices to guarantee, for any such inference relation, interpolation on the model-theoretic level.

A third part of the book concerns neighbourhoods. These are useful when there is a shortage of models that are 'ideal' in a desired respect, but there are still plenty of 'more or less ideal' models that can be bundled into sets to be used as surrogates for them; those sets are the 'neighbourhoods'. For example, in the preferential semantics for qualitative uncertain inference, a formula may have no minimal models but we can reformulate our definition of the consequence relation in terms of a suitably defined downwards-closed set of models satisfying

the formula. Unlike the preceding two parts of the text, there is no central theorem here, but rather a piecemeal exploration of situations in which such surrogates may be available and of how much mileage we can get out of them. As the authors say, the chapter "should be seen as a toolbox, where one finds the tools to construct the semantics one needs for the particular case at hand".

But what of the terms 'modularity' and 'conditionals' in the title of the book? It turns out that quite often that the model-theoretic condition for good syntactic behaviour is some kind of modularity or independence property, saying that the effect of applying a certain operation to a composite item may be obtained by first applying it to the components and then recomposing the outcomes. In some cases, this takes the form of a distribution principle. For example the 'multiplicative size rule' for $\mu$ mentioned above tells us that $\mu$ distributes over a natural two-place product operation: $\mu(M \times M') = \mu(M) \times \mu(M')$.

The use of the term 'conditional' is rather less transparent. The authors explain that "it seems best to say that a conditional is just any operator" (of any arity) over the powerset of all models for a given language—thus including for example negation and conjunction which, however, are left aside from the discussion because "we know them well"! So the theory of conditionals, as understood in this book, is really a general theory of operations in models that can serve as interpretations of connectives of a formal language.

This brief review does no justice to the development of subtle concepts, deep results, and ingenious proofs in the volume. But even a review several times longer could not do so. For the detail, there is no substitute for following through the chapters one by one. The task is rather arduous, as the network of definitions and theorems is dense. To help the reader, each chapter begins with gives a rough and intuitive outline of its contents. The first chapter also provides an introductory overview which, however, is essentially a cut-and-paste composition of the separate chapter outlines.

**David Makinson**
London School of Economics