Tim Button and Sean Walsh. *Philosophy and Model Theory*. Oxford University Press 2018. 544 pp. $105.00 USD (Hardcover ISBN 9780198790396); $34.95 USD (Paperback ISBN 9780198790402).

The book is the work of three authors, and divided into the main text and the appendix. Hodges's appendix, originally written in the 1990s, presents a historical view of model theory with the emphasis placed on topics that are interesting for a model theorist who may not be focusing on related philosophical questions. A reader might find it useful to look at the historical appendix first.

The main text comprises three parts, or in total, 17 chapters. The parts are devoted to topics (Reference and Realism, Categoricity, and Indiscernibility and Classification), which impact philosophers' opinions in the philosophy of mathematics and more widely, in the philosophy of sciences. Each part comes with a flowchart that shows the dependencies between sections; this is helpful if somebody intends to read only a selection from the book. Most chapters end with sections that serve as appendices presenting technical definitions and theorems (often with proofs). The book ends with a copious bibliography and three indices (of subjects, names, symbols, and definitions), making the book user-friendly.

The first chapter introduces first- and second-order languages, which are the most frequently used languages in the rest of the book. The authors distinguish between Tarskian, Robinsonian and hybrid semantics, which differ in handling name constants and are usually lumped together. Chapter 2 introduces the problem of reference in mathematics and how isomorphism of structures muddles the connection between terms and their referents. For starters, the problem of natural numbers is described in light of Benacerraf's, Putnam's, Shapiro's, and others' views. It might appear that replacing non-logical constants with quantified variables could resolve the problems with reference. In chapter 3, the authors explore this 'ramsified' realism, which impacts not only our thinking about math, but about other sciences too. Theoretical terms, which might be viewed as problematic, can be existentially quantified away, in effect, replacing them with suitable (theory-free) descriptions. This idea was pioneered by F. Ramsey, and later promoted in the work of R. Carnap; however, it is shown here that the ramsified theory, even with an intended model at hand, can guarantee nothing else but the least cardinal the class of theoretical objects must have. Chapter 4 introduces non-standard reals, infinitely large and infinitely small entities. Given a first-order theory (such as Peano arithmetic or the theory of reals), an intended model may be expanded with non-standard objects using the compactness of first-order logic. A. Robinson's development of non-standard analysis is an excellent example of how technical developments can affect philosophical views. Infinitesimals, which went from mysterious but useful quantities to outright abominable ones, have been readmitted into the class of reals. The non-standard reals are pressed into a model by a set of first-order sentences, and the resulting models are elementary extensions. Then, as is to be expected, the next chapter scrutinizes various ways (of which there are many) that may justify structures or theories to be called the same.

The second part of the book deals with the philosophical problems that arise from categoricity (or rather, the lack thereof). Chapter 6 lays out the problem: If mathematicians talk about structures that are the same up to isomorphism, then people should be able to select isomorphism types. Chapter 7 starts with the statement and informal explanation of the Löwenheim–Skolem theorems for first-order theories. An immediate consequence of these theorems is that no first-order theory that has infinite models is categorical. In particular, Peano arithmetic (PA) is not categorical, hence, often PA_2 is used instead, the full models of which (by Dedekind's result) are all isomorphic. However, second-order theories (e.g., PA_2 or ZFC_2) come with their own problems such as the nonexistence of a sound and complete proof system and the failure of compactness. Some weaker (than PA_2) alternatives are mentioned, but none of them stands out as the ideal theory to provide a categorical characterization of the natural numbers. Chapter 8 turns to another first-order theory, ZFC, the Zermelo–Fraenkel set theory with the axiom of choice. An intended model of ZFC may be the cumulative hierarchy, which the authors introduce together with regular and strongly inaccessible cardinals. Furthermore, we may insist that the model is transitive (i.e., that all the elements of the domain are also subsets of it, and true membership is modeled). Yet, Skolem's 'paradox' applies, because for every transitive model of ZF there is a countable transitive model that is elementarily equivalent (while the existence of a non-countable set is provable). It is possible to switch to a second-order theory; however, ZFC_2 may not be the best option. Unlike PA_2, ZFC_2 only provides a quasi-categoricity result (due to Zermelo). On another view, every model of ZFC_2 is isomorphic to a stage of the cumulative hierarchy indexed with an inaccessible cardinal. This leaves many stages of the cumulative hierarchy aside, which leads to the question of whether our concept of a set is as well captured by ZFC_2 as our notion of a number is captured by PA_2. If iteration (in set formation) is the basic operation that yields our concept of a set, then a theory that has every stage of the cumulative hierarchy as its model may be preferable. The second-order Scott–Potter level theory (SP_2) accomplishes this; SP_2 is not categorical, however, any of its models is isomorphic to a cumulative stage. The lack of categoricity for theories of fundamental concepts such as natural numbers and sets (discussed in the previous chapters) may lead to skepticism; somebody could claim that phrases normally used in mathematics (e.g., 'the natural numbers') do not refer to a particular structure, and users of such locutions do not possess the concepts (which are normally associated to them). In chapter 9, the skeptical views are discussed and dismissed in favor of an internalist approach. The latter is explicated in the next two chapters for natural numbers and for sets. An internalist rewrites the axioms of the respective theories (PA, SP, ZFC) in a second-order theory, and using only the deductive tools of SOL, obtains theorems that state (within the theory) the (quasi-)categoricity of the theory. The notion of an isomorphism is formalizable in SOL, which allows proofs of categoricity to be carried out—without any appeal to semantics or models. Given this, it may be surprising that the next chapter is devoted to internalist approaches to truth. However, the discussion leads to Boolean-valued models in chapter 13, which are used by the authors to show that there is an underdetermination with respect to the semantics of the languages even at the level of logical constants.

The last part of the book deals with four topics that have points of connection with philosophical debates. Chapter 14 introduces compactness from a topological point of view together with Stone's duality theorem. The latter has been extended to normal modal logics, and in that context the 'propositions are sets of possible worlds' and the 'possible worlds are maximal sets of sentences' viewpoints are two sides of a coin. An adage, which is often attributed to Leibniz, is that objects that cannot be distinguished are identical. The authors carefully examine this idea in chapter 15, and point out its dependence on the vocabulary and the expressive power of the logic, which results in a multitude of indiscernibility principles. As it turns out a definable linear order is the only obstacle to indiscernibility. Chapter 16 deals with generalized quantifiers, a class of quantifiers that bind variables in formulas and are interpreted through the extensions of the open formulas that are their arguments. The Härtig quantifier allows one to define an extension of PA that is weaker than PA_2 yet is categorical. Other generalized quantifiers are of interest in mathematical theories and in natural language semantics. The authors focus on the problem of 'logicality' along the lines of Tarski and Sher's criteria (permutation invariance and bijection invariance). The last chapter outlines the aims of Shelah's classification theory. Theories may have few or many uncountable models. The main gap theorem (in informal terms) says that in the former case, the models of the theory can be classified by 'cardinality-like' invariants, whereas in the latter case, they are unclassifiable. Even a glimpse at the latest developments in this area may divulge a contrast between some of the philosophical hesitation concerning the acceptance of (certain) abstract structures and the actual handling of such structures in mathematical practice.

This outstanding book collects a formidable selection of logical results (many of them with proofs) and it shows how the logical results sway various philosophical debates. The authors mainly limit their considerations to theories in two-valued logic, which is the bulk of mathematical theories. Hopefully, this book will inspire similar investigations into connections between philosophy and the model theory of other logics, such as intuitionistic, modal and relevance logics, including formal theories based on these logics.

Katalin Bimbó, University of Alberta