Alan Weir
Truth Through Proof: A Formalist Foundation for Mathematics.
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In this fascinating book, Weir defends a new account of what makes mathematical assertions objectively true or false. Roughly, they are true if there is a concrete proof of them, false otherwise. *Prima facie*, this account is hopeless. How, for example, could there be enough *concrete* proofs to establish even a fraction of mathematical truths? And didn't Gödel conclusively establish that mathematical truth outstrips mathematical proof? Weir maintains that by invoking an appropriate notion of idealization and giving up on our unfounded prejudice against infinitary proofs, his account can be made to work. This book is one long, and sometimes complex, argument to this effect.

Given the sheer number of controversial stances that go into making Weir's argument plausible, it is unlikely that many will be convinced. Moreover, it is unfortunate that his account of idealization is not as clear as it could be and that, occasionally, it is hard to follow his overall argument because he explores tangential issues in considerable detail. Yet, despite these flaws, this book is well worth reading for at least two reasons: it is fascinating to explore the kinds of moves one might want or have to make if one is tempted by Weir's general strategy. Also, many of Weir's challenges to orthodox perspectives have merit or interest in their own right.

Let's take a look at some details. In theorizing about a particular branch of mathematics, Weir maintains that we should distinguish between three levels: a) the *Game* or *G-level*, which consists of the assertion and manipulation of mathematical symbols strings (or similar) according to certain rules—think, for example, of a child performing long division using her favored procedure—b) the *Contentful* or *C-level*, which consists of semantically interpretable assertions using correlates of the aforementioned G-level symbol strings—for example, the aforementioned child might assert "208035 divided by 69 is equal to 3015"—and c) the *Metaphysical* or *M-level*, which is metatheoretic and includes formal specifications of G- and C-level languages and the system of G-level rules—the G-level proof theory—as well as observations that link the truth or falsity of C-level assertions to the provability or refutability of their G-level correlates.

Most of this book is devoted to developing this basic neo-formalist perspective, though it also includes arguments that this perspective is superior to various alternatives. Of particular note are Chapter 5's discussion of the applicability of mathematics, Chapter 6's discussion of how to ground proof in the concrete, Chapter 7's discussion of idealization, which is intended to allow Weir to both reject strict finitism and meaningfully criticize finitism concerning proof, and Chapter 8's discussion of logic and how to recapture most of everyday mathematics within Weir's framework.

Weir's treatment of mathematics' applicability is inspired by Hartry Field's (1980), but Weir's account adds a few important wrinkles. First, he is a realist about physical properties and relations, which, arguably, makes the challenging task of "nominalizing" empirical theories easier for him than for Field. Second, mixed sentences are literally true or false since they are made true or false by a combination of the world, proof-theoretic facts, and stipulated "bridge principles". Third, while mathematics is "ontologically dispensable" to empirical science, it is not "conceptually dispensable"; that is, the contents of many empirical theories cannot be expressed without using mathematics. Fourth, an important part of the reason why mathematics is "conceptually indispensable" to empirical science is that frequently, when empirical scientists express their theories using mathematics, they engage in *injective idealization*; that is, idealization into but not onto a given domain. Weir illustrates this notion with the following example:

> … suppose one thought that space-time was really finite, a finite matrix of atomic cells let us say. Even so, one might still wish to use a four-dimensional Riemannian manifold to represent it because of the convenience of using analytical techniques such as differentiation and integration in accounting for mechanical and dynamical phenomena. (137)

This notion of injective idealization also plays a key role in Weir's account of provability. According to that account, a G-level mathematical assertion from a given mathematical practice is provable, and hence its C-level correlate true, if it is provable in a formal system that is a legitimate injective idealization of the practice in question, while such a G-level assertion is refutable if it is refutable in such a formal system. Hence, provability and refutability are really features of formal systems and only apply to concrete tokens, since they can be "pulled back" from these formal systems by inverting the relevant injective maps from the germane practices to the formal systems in question. Accordingly, some G-level assertions are provable not in virtue of the existence of a concrete proof token, but in virtue of being provable in a formal system. While *prima facie* this involves Weir in ontological commitments to abstract proofs, he argues that the relevant formal systems are pieces of applied mathematics and, as such, are free of such commitments. To my mind, however, Weir's argument in Chapter 5 (to the effect that applied mathematical theories are free of ontological commitments to abstract entities) works only if such theories are made true or false by a combination of the world and *concrete* proof tokens—that is, if mathematical truth and falsity genuinely "bottom out" in concrete proof or refutability. But this is not so, according to Weir's account.

This issue aside, neo-formalism still faces Gödelian worries, which is to say: so long as theoremhood in the relevant formal systems is recursively enumerable, mathematical truth will outstrip formal proof. But, Weir asks, why should we restrict ourselves to formal systems of this type? Why not consider infinitary systems in which theoremhood in not recursively enumerable? The traditional answer is epistemological: we could not hope to be convinced by anything but a finite proof. Yet an extremely large finite proof, Weir argues, does no more for us epistemically than an infinitary proof: the powers of both to convince rely in significant ways on idealization. Moreover, what is needed to *establish* the truth or falsity of a given C-level mathematical assertion, S, according to neo-formalists, is a *concrete*, and hence *finite*, proof to some effect. More specifically, what is required is *either* that there exist a concrete proof of S in a formal system that is a legitimate injective idealization of the relevant practice *or* that there exists a concrete proof to the effect that S is provable in such a formal system. Thus, even if neo-formalists accept infinitary formal systems as legitimate injective idealizations of concrete mathematical practices, it is concrete, finite, proofs that do the real epistemic work for them. Therefore, there is no epistemic problem with neo-formalists appealing to such systems. Further, from a technical perspective, by appealing to infinitary formal systems to which Gödel's incompleteness theorems do not apply, neo-formalists clearly overcome Gödelian worries. Indeed, Weir's actual response to these worries is to argue that the aforementioned types of infinitary systems really can and do serve as legitimate injective idealizations of concrete mathematical practices. So, Weir's solution to the *prima facie* problems with his neo-formalist strategy is to idealize mathematical languages, where the legitimate injective idealizations in question can be infinitary.

This is a lot of work for the notion of a legitimate injective idealization to perform; it is unclear that it can perform this work. Even worse for Weir is that it is unclear that he may rightfully appeal to this notion in the way that he does. To see why, consider what makes a formal system a legitimate injective idealization of a concrete practice, P. Weir is never overly clear, but I do not see how such a system could stand in this relation to P without P having some semantic interpretation. Perhaps, as Weir intends, a single such system could be taken to specify a semantic interpretation for P in some syntactic sense. Yet Weir believes that more than one formal system may be a legitimate injective idealization of the same practice. But why, then, do two such idealizations of P not simply amount to two *different* interpretations of P? If they don't, mustn't P have some *independent interpretation* against which both idealizations can be assessed to be legitimate? Yet any such independent interpretation of P would determine truth and falsity in P *independently of* the syntactic semantics that Weir seeks to provide for it. Thus, it would seem, the notion of a legitimate injective idealization of a concrete mathematical practice cannot perform the role that Weir intends for it.

This is not the only problem with Weir's neo-formalism, however. Frege famously maintained that what raises mathematics—well, arithmetic—from the rank of a game to that of a science, i.e., a discourse whose assertions are true or false, is its applicability. Weir, I fear, has failed to fully appreciate the force of Frege's insight. While he has understood it sufficiently to realize that neo-formalists must provide an account of mathematics' applicability, he has not understood it sufficiently to realize that mathematics' applicability has to be an integral part of what makes its assertions true or false. In an important sense, Frege's insight is that mathematical assertions obtain their truth-evaluability from their applicability. Yet, according to Weir, mathematical systems develop as uninterpreted formal calculi that are only later recognized to be applicable in various settings.

Weir, of course, might take Frege's so-called insight to be no insight at all, and take his arguments against Platonism to demonstrate this. Yet Frege's insight is separable from his Platonism. To appreciate this, let us consider Weir's understanding of games. He, unlike his game formalist predecessors, recognizes that games are meaningful activities. Yet he assigns them a mere "internal" meaningfulness; roughly, they are meaningful in virtue of being activities aimed at particular, internal, goals (e.g., checkmating one's opponent). But the meaningfulness of games goes well beyond such internal standards. Games, like all institutions, also perform various external functions, and, in so doing, inherit a kind of external meaningfulness (e.g., many games serve as non-fatal ways of assessing the relative merits of two individuals or groups of individuals in some respect). Frege's real insight is that one can only fully appreciate mathematics if one recognizes that it, too, is meaningful in this external sense, that is, if one recognizes that particular branches of mathematics are introduced to serve particular functions. Moreover, once one appreciates the functions that particular branches of mathematics perform, one finds that those functions typically constrain the subject matters of those respective branches sufficiently to determine the truth values of mathematical assertions within them.

In summary, then, there is little chance that Weir's neo-formalism is correct; it assigns proof a role in mathematics that it is unlikely to be able to perform. Despite this, this book is worth reading because many of Weir's challenges to orthodox perspectives are insightful, and there is value in exploring Weir's perspective on mathematics, if only to appreciate why it is incorrect.

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