Feng Ye
Strict Finitism and the Logic of Mathematical Applications.
Springer Science 2011.
272 pages
$159.00 (Hardcover ISBN 9789400713468)

A well-known worn-out criticism of constructive mathematics generally is that it is too weak to produce enough mathematics to be of use. This undeserved criticism has been debunked, time and again. This book presents another contribution to that debunking. However the real contribution of the book takes this idea a (big) step further. As the title suggests, it demonstrates that the lion's share (perhaps even essentially all) of the mathematics that is in use today in the physical sciences – up to and including quantum mechanics and general relativity – can be handled using strict finitist constructive methods. The strict finitist method used in the book is essentially quantifier-free primitive recursive arithmetic, and the resulting mathematics is essentially elementary recursive mathematics without quantifiers. That such inroads could be made into modern mathematics using such a minimal framework should come as no surprise (since Russian recursive mathematics is essentially Bishop-style mathematics (BISH) together with a version of Church's thesis and Markov's principle of unbounded search), but what is perhaps surprising is the fact that no notion of infinity (potential or otherwise) is used at all in the book.

The questions the monograph focuses on [5, p. v] are what level of commitment to the notion of infinity is necessary to imply scientific conclusions about the actual universe, whether applied mathematical models that use infinities derive literal truths about finite phenomena, and how, if at all, the use of infinities in the mathematics makes life for the scientist easier.

Ye's approach is motivated in the first chapter, which is an expanded version of [6]. He briefly outlines his philosophical position, which he calls *radical naturalism*. Appropriately, he neither defends his position nor refutes alternatives; the book's focus is to develop mathematics, rather than a philosophy. Despite the name he gives it, the stance he takes throughout the book can hardly be called radical. Perhaps *minimally committal* is more appropriate. Regarding the mathematics, the reader need not buy into Ye's philosophical position. On the contrary, the presentation of material is understandable (and relevant) within a wide variety of positions. In this, Ye's presentation is much like Bishop's work – able to be read and understood by the working mathematician and physicist of almost any persuasion.

As a whole, the book tries to offer insight into how mathematics involving infinities is applicable for deriving truths about finite things, and, further, hopes to give understanding about the logic behind mathematical applications, and what the nature of mathematics is. The first chapter finishes with an example of how, in the author's view, a scientist using the mathematical apparatus of differential equations concerning continuous things is able to derive literal truths about discrete phenomena. The contribution of the monograph is to side-step the logical problem of how a mathematical premise that is not literally true may be used to derive literal truths.

Ye sees this work as a scientific study of the success of applied mathematics as a natural phenomenon. The upshot of Ye's philosophical position, if he is correct, is that it solves a logical puzzle – it allows us to deduce true things about real spacetime (within modern theoretical physics), regardless of whether it is continuous or discrete on a microscopic level.

The logical commitments of Ye's strict finitism are laid down in Chapter 2. The formal system for strict finitism (SF) is technically detailed, and is essentially a typed λ-calculus, with a few constants and operators to define basic objects and operations. Of particular note is Lemma 2.7, where the author establishes that any mathematical theorem in the book that is established with induction, can be established without induction (i.e. has a proof containing only closed formulae). On the face of it, this is a fairly innocuous claim; however it forces a conception of implication (detailed later in Section 2.2.2) which restricts the notion of 'arbitrary proof' as normally given in intuitionism via the BHK interpretation. In effect, Ye uses Bishop's *numerical implication* [5, p. 55]. The upside of this version of implication is that it starts from a hypothetical witness (rather than an arbitrary proof) and allows using finitely many instances to prove generalized conclusions. Arguably, a downside of this is that the axiom of choice is then *provable* in SF. Here is where the philosophical commitments of Ye's views impact upon the mathematics, and somewhat narrow the scope for application within the wider audience. It also allows one to (almost) recover the classical duality between the quantifiers, when understood in Ye's sense. (It would perhaps be interesting to see if the principle of excluded middle follows, over SF, from the axiom of choice in that setting. I suspect not.) Further, some statements which are typically not intuitionistically provable become provable in this setting. For example, with suitable definitions of ordering, etc., the statement

$$\neg x \leq y \leftrightarrow x > y$$

is provable here (though, as in intuitionism, $x \leq y$ does not imply $x < y \vee x = y$).

Ye's 'starred' notation is strongly reminiscent of a translation, and indeed he mentions Gödel's translation of classical logic for the intuitionist. Classical theorems can be translated into SF claims; if these claims are subsequently proven, this 'assigns numerical content to a classical theorem' [5, p. 62].

Of particular note is Theorem 2.11 [5, p. 56], where Ye shows that all the logical operators of SF follow the laws of intuitionistic logic, as well as the axiom of choice. Since dependent choice is a consequence of choice, and BISH is mathematics performed with intuitionistic logic and dependent choice, a corollary of this result is that *all the results of Bishop & Bridges* [1, and subsequent works in BISH] *hold in SF*. In other words, there are no examples of theorems that can be established in BISH which cannot be established in SF. (To put this in the terminology of Bishop-style mathematicians, strict finitism is an interpretation of Bishop-style constructive mathematics, with additional assumptions/philosophical commitments. BISH is consistent with SF, just as it is with classical mathematics, Brouwer's intuitionistic mathematics, and (Russian) recursive mathematics.)

Chapters 3-7 closely follow the work of Bishop & Bridges [1]. The difference here largely lies in the fixing of foundation in primitive recursive functions: extra effort is needed to show that certain constructions are bounded in the right way, and that any inductions used are reducible to quantifier-free induction. In Ch. 3, the usual sorts of results obtain: the classical least upper bound principle for the real numbers is invalid, but with stronger (and classically trivial) hypotheses a useful version of the principle obtains, as well as a version of the intermediate value theorem.

Differentiability and integration of real functions is treated in the expected recursive manner, with a few modifications to ensure no unbounded searches occur. Some emphasis (at least a token amount) is placed on applications, with examples thrown in to show that strict finitism is able to support most of the work in modern scientific theories.

Metric spaces (Ch. 4), complex analysis (Ch. 5), and integration (Ch. 6) are all laid out in the development of the book, largely following the Bishop-style development. Comments, such as "We will verify that the constructive proof of [the Stone-Weierstrass] theorem [in [1]] is available to strict finitism" [5, p. 121] perhaps suggest that there is a worry that some results in BISH are not available to SF. In light of Theorem 2.11 (and the discussion above), there need be no such worry. To the author's credit, however, he explicitly shows that the proofs from BISH may be moved into SF, sometimes with a great deal of double-checking and changes required to fit into the SF framework (see, for example, Ch. 6, where great pains have been taken to ensure that everything in sight is able to be done with quantifier-free induction and bounded searches).

Chapter 7 develops the theory of linear operators on Hilbert space. The author emphasizes that the definition of linear space must be modified from the Bishop framework in order to fit into SF. The definition provided [5, Def. 7.1] satisfies all the properties of that in [1, Def. 7.1.1], and (if the domain of applicability of BISH is confined to that of SF) vice versa. However, the author develops the theory of unbounded operators on linear subsets of a Hilbert space [5, Section 7.5 onwards], and establishes a spectral theorem for unbounded self-adjoint operators; this is not found in [1], where only bounded operators are considered. (However, see also [2], for some work in this area, as well as Bas Spitters' work on constructive operator theory. In particular, in [4], constructive conditions are established for spectral decomposition of certain unbounded operators.) This is in order to establish Stone's theorem. The author's motivation (and the painstaking work of the earlier chapters) becomes vindicated from here on. This chapter establishes enough Hilbert space theory to formalize much of modern quantum mechanics.

Chapter 8 is new work on semi-Riemannian geometry, supporting large parts of the general theory of relativity. Together with the work from Chapter 7, this forms the prize of the book: most of the mathematics necessary for modern theoretical physics can be developed within a strict finitist framework. In Chapter 8, Ye outlines semi-Riemannian geometry sufficient for proving a version of Hawking's singularity theorem. (In order to do so, Ye does use a mild assumption – called the Geodesic Stability Assumption in the text – which is automatically true in the classical framework, and so the author commits to no more – classically – than the classical theorist.) This directly refutes claims from some critics that the constructive theory is insufficient to address fundamental problems in general relativity; Hellman, for example, specifically claims that "the singularity theorems of Hawking and Penrose … appear to elude constructivization." [3, p. 425].

All in all, there is something in this book for most of the intended audience: the philosopher will find a rigorous presentation of strict finitism and its consequences for scientific theories (which seem difficult to argue against, given the success of the theory); the mathematician will discover that the idealized notion of infinity is not necessary for developing current scientific theory (and so we do not need to assume it); and the logician will find that recursive considerations are sufficient for supporting not only extended mathematical reasoning, but conclusions about the real world.

References

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